Nonperturbative renomalization group for Einstein gravity with matter

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February 1, 2008

Abstract

we investigate the exact renormalization group (RG) in Einstein gravity coupled to N-component scalar field, working in the effective average action formalism and background field method. The truncated evolution equation is obtained for the Newtonian and cosmological constants. We have shown that screening or antiscreening behaviour of the gravitational coupling depends cricially on the choice of scalar-gravitational ξ and the number of scalar fields.

There has recently been much activity in the study of nonperturvative RG dynamics in field theory models (for recent references, see [1]). One of the versions of nonpertubative RG based on the effective average action has been developed in ref.[2] for Einstein gravity (the gauge dependence problem in this formalism has been studied in ref. [3]) and in [4] for gauged supergravity. The nonperturbative RG equation for cosmological and Newtonian coupling constants have been obtained in ref.[2] for the Einstein gravity and in ref [5] for R^2 gravity. The comparison between quantum correction to Newtonian coupling from nonperturvative RG [2] and from effective field theory technique [6] has been done.

It is quite interesting to study nonperturbative RG (or evolution equation) for the gravity coupled to N-component scalar field in orden to evaliate the influence of the scalar coupling and the number of scalar in the nonperturbative behaviour of the Newtonian and cosmological constants. We follow the formalism of ref. [2] developed for gravitational theories. The basic elements of it are the background field method (see [7] for a review) and the truncated nonperturbative evolution equation for the effective average action [2]. We will start from the theory with the following action.

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi\bar{G}}\big(-R+2\lambda\big)+\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi_i+\frac{1}{2}\xi R\phi^i\phi_i\Big], \qquad (1)$$

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where $i =, ..., N$ and nonperturbative Einstein gravity is considered to be valid below some UV scale Λ. We will use the truncation [2], [3]

$$\bar{G} \to G_k = \frac{\bar{G}}{Z_{Nk}}, \lambda \to \lambda_k \tag{2}$$

Following the approach of ref.[2] we will write the evolution equation for the effective average action $\Gamma_k[g, \bar{g}]$ defined at non zero momentum ultraviolet k below some cut-off $\Lambda_{cut-off}$.
The truncated form of such evolution equation is

$$\begin{aligned}\partial_t \Gamma_k[g, \bar{g}] =& \frac{1}{2} Tr[(\Gamma_k^{(2)}[g, \bar{g}] + R_k^{grav}[\bar{g}])^{-1} \partial_t R_k^{grav}[\bar{g}]] \\ & - Tr[(-M[g, \bar{g}] + R_k^{gh}[\bar{g}]) \partial_t R_k^{gh}[\bar{g}]],\end{aligned} \tag{3}$$

where $t = Lnk, k$ is the nonzero momentum scale, R_k are cutt-off, $M[g, \bar{g}]$ are ghost o perators, $\bar{g}_{\mu\nu}$ is the background metric, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the quantum field. $\Gamma_k^{(2)}$ is the Hessian of $\Gamma_k[g, \bar{g}]$ with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$ (for more details, see[2]).
Projecting the evolution equation on the space with low-derivatives terms, one gets the left-hand side of the truncated evolution equation (3) as follows:

$$\partial_t \Gamma_k[g, \bar{g}] = 2k^2 \int d^4x \sqrt{g} [-R(g) \partial_t Z_{Nk} + 2 \partial_t (Z_{Nk} \lambda_k)], \tag{4}$$

with $k^2 = \frac{1}{32\pi \bar{G}}$. The initial conditions for Z_{Nk}, λ_k are chosen in the same way as in [2].
In the right-hand side of the evolution equation (2) we need the second functional derivate of $\Gamma_k[g, \bar{g}]$ at fixed background $\bar{g}_{\mu\nu}$,

$$\begin{aligned}\Gamma_k^{(2)}[g, g] = Z_{Nk} k^2 \int d^4x \sqrt{g} \Big\{ & \frac{1}{2} \hat{h}_{\mu\nu} \big[-\Box - 2\bar{\lambda}_k + R \big] \hat{h}_{\mu\nu} - \frac{1}{8} \phi [-\Box - 2\bar{\lambda}_k] \phi \\ & + \frac{1}{2} \tilde{\phi}_i [-\Box + \xi R] \tilde{\phi}^i - R_{\mu\nu} h^{\nu\rho} h_\rho^\mu + R_{\alpha\beta\mu\nu} h^{\beta\nu} h^{\alpha\mu} \Big\}\end{aligned} \tag{5}$$

where we have rescaled the scalar field

$$\phi_i \to \tilde{\phi}_i = 32\pi \bar{G} \phi_i. \tag{6}$$

In orden to calculate the effective average action, we have to specify some curve background.
A useful choice is a maximal symmetric space where the curvature R is the external parameter characterizing the space. In such space we have

$$\begin{aligned}\Gamma_k^{(2)}[g, g] = Z_{Nk} k^2 \int d^4x \sqrt{g} \Big\{ & \frac{1}{2} \hat{h}_{\mu\nu} \big[-\Box - 2\bar{\lambda}_k + \frac{2}{3} R \big] \hat{h}_{\mu\nu} \\ & - \frac{1}{8} \phi [-\Box - 2\bar{\lambda}_k] \phi + \frac{1}{2} \tilde{\phi}_i [-\Box + \xi R] \tilde{\phi}^i \Big\}\end{aligned} \tag{7}$$

The complete average effective action wiht $\bar{g} = g$ containing gravitational, ghosts and scalar contributions is calculated with final result

$$
\begin{aligned}
\bar{\Gamma}_k = \frac{1}{2} Tr_T Ln\Big[Z_{Nk}\Big(-\Box - 2\lambda_k + \frac{2}{3}R + k^2 R^{(0)}(-\Box/k^2)\Big)\Big] \\
+ \frac{1}{2} Tr_S Ln\Big[Z_{Nk}(-\Box - 2\lambda_k + k^2 R^0(-\Box/k^2))\Big] \\
- Tr_V Ln[(-\Box - \frac{1}{4}R + k^2 R^{(0)}(-\Box/k^2))] \\
+ \frac{N}{2} Tr_S Ln[Z_{Nk}(-\Box + \xi R + k^2 R^{(0)}(-\Box/k^2))]
\end{aligned}
\tag{8}
$$

Now we want to find the RHS of the evolution equation . To this end, we differentiate the average action (6) with respect to t. Then we expand the operators in (6) with respect to the curvature R because we are only interested in terms of orden $\int d^4x\sqrt{g}$ and $\int d^4x\sqrt{g}R$:

$$
\begin{aligned}
\partial_t \Gamma_k[g,g] = & Tr_T\Big[\mathcal{N}\Big(A + \frac{2}{3}R\Big)^{-1}\Big] + Tr_S[\mathcal{N}A^{-1}] \\
& - 2Tr_V\Big[\mathcal{N}_0\Big(\mathcal{A}_0 - \frac{1}{4}R\Big)^{-1}\Big] + NTr_S[\mathcal{N}_0(\mathcal{A}_0 + \xi R)^{-1}],
\end{aligned}
\tag{9}
$$

where

$$
\mathcal{N} = \frac{\partial_t[Z_{Nk}k^2 R^{(0)}(z)]}{Z_{Nk}},
\tag{10}
$$

$$
\mathcal{A} = -\Box + k^2 R^{(0)}(z)] - 2\lambda_k.
\tag{11}
$$

The operators $\mathcal{N}_0$ and $\mathcal{A}_0$ are defined from (8) with $\lambda_k = 0$ and $Z_{Nk} = 1$ (see [2], [3]). Here the variable z replaces $-\Box/k^2$ and $\eta_N(k) = -\partial_t(LnZ_{Nk})$. Note that as a cut-off we use the same function as in ref.[2]: $R^{(0)}(z) = \frac{z}{exp[z]-1}$. The above steps lead then to

$$
\begin{aligned}
\partial_t \Gamma_k[g,g] = & Tr_T\Big[\mathcal{N}A^{-1}\Big] + Tr_S[\mathcal{N}A^{-1}] - 2Tr_V\Big[\mathcal{N}_0\mathcal{A}_0^{-1}\Big] \\
& + NTr_S[\mathcal{N}_0\mathcal{A}_0^{-1}] - R\Big\{\frac{2}{3}Tr_T[\mathcal{N}\mathcal{A}^{-1}] + \frac{1}{2}Tr_V[\mathcal{N}_0\mathcal{A}_0^{-2}] \\
& + \xi NTr_S[\mathcal{N}_0\mathcal{A}_0^{-1}]\Big\} + O(R^2),
\end{aligned}
\tag{12}
$$

As a next step we evaluate the traces. We use the heat kernel expansion which

for an arbitrary function of the covariant Laplacian $W(D^2)$ reads

$$\begin{aligned} Tr_j[W(-D^2)] =&(4\pi)^{-1} tr_j(I)\Bigg\{ Q_2[W]\int d^4x\sqrt{g} \\ &+\frac{1}{6}Q_1[W]\int d^4x\sqrt{g}R + O(R^2)\Bigg\}, \end{aligned} \tag{13}$$

where by I we denote the unit matrix in the space of field on which D^2 acts. Therefore $tr_j(I)$ simply counts the number of independent degrees freedom of the field. The sort j of fields enters (15) via $tr_j(I)$ only. Therefore, we will drop the index j after the evaluation of the traces in the heat kernel expansion.
The functionals Q_n are the Mellin transforms of W,

$$Q_n[W] = \frac{1}{\Gamma_{(n)}}\int_0^\infty dz z^{n-1} W(z), \qquad (n>0) \tag{14}$$

Now we have it perform the heat kernel expansion (10) in eq.(9). This leads to a polynomial in R which is the RHS of the evolution equation (2).
By the comparison of coefficients with the LHS of the evolution equation (3), we obtain at the orden $\int d^4x\sqrt{g}$

$$\partial_t(Z_{Nk}\bar{\lambda}_k) = \frac{1}{4k^2}\frac{1}{(4\pi)^2}\{10Q_2[\mathcal{N}/\mathcal{A}] - 8Q_2[\mathcal{N}_0/\mathcal{A}_0] + NQ_2[\mathcal{N}_0/\mathcal{A}_0]\} \tag{15}$$

and at the orden $\int d^4x\sqrt{g}R$.

$$\begin{aligned} \partial_t Z_{Nk} =& -\frac{1}{12k^2}\frac{1}{(4\pi)^2}\{10Q_1[\mathcal{N}/\mathcal{A}] - 8Q_1[\mathcal{N}_0/\mathcal{A}_0] \\ &+ NQ_1[\mathcal{N}_0/\mathcal{A}_0] - 36Q_2[\mathcal{N}/\mathcal{A}^2] - 12Q_2[\mathcal{N}_0/\mathcal{A}_0^2] - 6\xi NQ_2[\mathcal{N}_0/\mathcal{A}_0^2]\}. \end{aligned} \tag{16}$$

The cutt-off-dependent integrals are defined in [4]

$$\Phi_n^P(w) = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z) - zR^{(0)'}(z)}{[z+R^{(0)}(z)+w]^P}, \tag{17}$$

$$\tilde{\Phi}_n^P(w) = \frac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^P}, \tag{18}$$

for $n>0$. It follows that $\Phi_0^P(w) = \tilde{\Phi}_0^P(w) = (1+w)^{-P}$ for $n=0$. We can rewrite eqs. (12) and (13) in terms of Φ and $\tilde{\Phi}$. This leads to the following system of equations:

$$\begin{aligned} \partial_t(Z_{Nk}\bar{\lambda}_k) =& \frac{1}{4k^2}\frac{1}{(4\pi)^2}k^4\{10\varphi_2^1(-2\bar{\lambda}_k/k^2) - 8\Phi_2^1(0) \\ &+ N\Phi_2^1(0) - 5\eta_N(k)\tilde{\Phi}_2^1(-2\bar{\lambda}_k/k^2)\}, \end{aligned} \tag{19}$$

$$\begin{aligned}\partial_t Z_{Nk} = &-\frac{1}{12k^2}\frac{1}{(4\pi)^2}k^2\{10\Phi_1^1(-2\bar{\lambda}_k/k^2)+(N-8)\Phi_1^1(0)\\&-36\Phi_2^2(-2\bar{\lambda}_k/k^2)-(12+6\xi N)\Phi_2^2(0)\\&-5\eta_N(k)\tilde{\Phi}_1^1(-2\bar{\lambda}_k/k^2)+18\eta_N(k)\tilde{\Phi}_2^2(-2\bar{\lambda}_k/k^2)\}.\end{aligned}\tag{20}$$

Now we introduce the dimensionless, renormalized Newtonian constant and cosmological constant

$$g_k = k^2 G_k = k^2 Z_{Nk}^{-1}\bar{G}, \qquad \lambda_k = k^{-2}\bar{\lambda}_k. \tag{21}$$

Here G_k is the renormalized Newtonian constant at scale k. The evolution equation for g_k reads then

$$\partial_t g_k = [2+\eta_N(k)]g_k. \tag{22}$$

from (16) we find the anomalous dimension $\eta_N(k)$

$$\eta_N(k) = g_k B_1(\lambda_k) + \eta_N(k) g_k B_2(\lambda_k) \tag{23}$$

where

$$\begin{cases} B_1(\lambda_k) = \frac{1}{6\pi}[10\Phi_1^1(-2\lambda_k)+(N-8)\Phi_1^1(0)-36\Phi_2^2(-2\lambda_k)-(12+6\xi N)\Phi_2^2(0)],\\ B_2(\lambda_k) = \frac{1}{6\pi}[18\tilde{\Phi}_2^2(-2\lambda_k)-5\tilde{\Phi}_1^1(-2\lambda_k)].\end{cases}\tag{24}$$

Solving (19)

$$\eta_N(k) = \frac{g_k B_1(\lambda_k)}{1-g_k B_2(\lambda_k)} \tag{25}$$

we see that the anomalous dimension η_N is a nonperturbative quantity. From (15) we obtain the evolution equation for the cosmological constant

$$\begin{aligned}\partial_t(\lambda_k) = &-[2-\eta_N(k)]\lambda_k + \frac{1}{2\pi}g_k[10\Phi_2^1(-2\lambda_k)+\\&+(N-8)\Phi_2^1(0)-5\eta_N(k)\tilde{\Phi}_2^1(-2\lambda_k)].\end{aligned}\tag{26}$$

Equations (18) and (22) together with (20) give the system of differential aquations for two k-dependent constants λ_k and g_k. These equations determine the value of the running Newtonian constant and cosmological constant at the scale $k \ll \Lambda_{cut-off}$. The above evolution equations include nonperturbative effects which go beyond a simple one-loop calculation.
Next, we estimate the qualitative behaviour of the running Newtonian constant as the above system of RG equations is too complicated and cannot be solve analytically. To this end we assume that the cosmological constant is much

smaller than the IR cut-off scale, $\lambda_k \ll k^2$, so we can put $\lambda_k = 0$ that simplifies eqs. (18) and (20). After that, we make an expansion in powers of $(\bar{G}k^2)^{-1}$ keeping oly the first term (*i,e.* we evaluate the functions $\Phi^P_n(0)$) and $\tilde{\Phi}^P_n(0)$) and finally obtain (with $g_k \sim k^2\bar{G}$)

$$G_k = G_0[1 - w\bar{G}k^2 + ...] \tag{27}$$

where

$$\begin{aligned} w &= \frac{1}{12\pi}[(48 + 6\xi N)\Phi^2_2(0) - (2 + N)\Phi^1_1(0)] \\ &= \frac{1}{12\pi}\Big[(48 + 6\xi N) - (2 + N)\frac{\pi^2}{6}\Big], \end{aligned} \tag{28}$$

where $\Phi^1_1(0) = \pi^2/6$ and $\Phi^2_2(0) = 1$.
In the case of Einstein gravity, a similar solution has been obtained in refs. [2], [3]. From (19) we can write the inequality

$$\Big[\frac{4}{\pi} - \frac{\pi}{36}\Big] + N\Big[\frac{\xi}{2\pi} - \frac{\pi}{72}\Big] > 0, \tag{29}$$

which means that the limit $N \to \infty$ and for values of $\xi > \pi^2/36$, the coefficient $w > 0$, which indicates that the Newtonian coupling decreases as k^2 increases; *i,e.* we find that the gravitational coupling is antiscreening. For $\xi < \pi^2/36$ we have $w < 0$ and a screening effect follows. For small N the dependence of the sign of w on ξ is given by eq. (24). Hence, we proved that screening or antiscreening behaviour of the gravitational constant depends crucially on the choice of ξ (which may be dictated by asymptotic freedom [7]) or the number of fields. That may lead to interesting coslogical consequences in the early universe.

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This work was supported by *COLCIENCIAS* (Colombia) Project N0.1106-05-393-95.
I thank S. D. Odintsov for stimulating discussions.

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